Medium–induced QCD cascade: democratic branching and wave turbulence

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We study the average properties of the cascade of gluons that is generated by an energetic parton propagating through a quark-gluon plasma. We focus on the soft, medium-induced, emissions which control the energy transport at large angles with respect to the leading parton. We show that the effect of multiple branchings are important. In contrast to what happens in a usual QCD cascade in vacuum, medium-induced branchings are quasi-democratic, with offspring gluons carrying sizable fractions of the energy of their parent gluon. This results in a new mechanism for the transport of energy towards the medium, which is akin to wave turbulence with a scaling spectrum $\sim 1/\sqrt{\omega}$. We argue that the turbulent flow may be responsible for the excess energy carried by very soft quanta, as revealed by the analysis of the di-jet asymmetry observed in Pb-Pb collisions at the LHC.

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One important phenomenon discovered recently in heavy ion experiments at the LHC is that of *di–jet asymmetry*, a strong imbalance between the energies of two back–to–back jets. This asymmetry is commonly attributed to the effect of the interactions of one of the two jets with the hot QCD matter that it traverses, while the other leaves the system unaffected. Originally identified [1, 2] as missing energy, this phenomenon has been subsequently shown [3] to consist in the transport of a sizable part of the jet energy by soft particles towards large angles. Some of the features of in-medium jet propagation are well accounted for by the BDMPSZ mechanism for medium-induced radiation (from Baier, Dokshitzer, Mueller, Peigné, Schiff [4] and Zakharov [5]). However, most studies within this approach have focused on the energy lost by the leading particle, while the LHC data call for a more thorough analysis of the jet shape for which the effects of multiple branching at large angles are important. Within that context, an important step was achieved in Ref. [6], where it was shown that, in a leading order approximation, one could consider successive gluon emissions as independent of each other. This allows one to treat multiple emissions as a probabilistic branching process, in which the BDMPSZ spectrum plays the role of the elementary branching rate [7–9].

Specifically, the differential probability per unit time and per unit z for a gluon with energy ω to split into two gluons with energy fractions respectively z and $1-z$ reads

$$\frac{\mathrm{d}^2\mathcal{P}_{\rm br}}{\mathrm{d}z\,\mathrm{d}t} = \frac{\alpha_s}{2\pi}\,\frac{P_{g\to g}(z)}{\tau_{\rm br}(z,\omega)}, \quad \tau_{\rm br} = \sqrt{\frac{z(1-z)\omega}{\hat{q}_{\rm eff}}}, \tag{1}$$

where $P_{g\to g}(z) = N_c\left[\frac{1}{z(1-z)} + z(1-z) - 2\right]$ is the leading order gluon-gluon splitting function, $\hat{q}_{\rm eff} \equiv \hat{q}\,[1-z(1-z)]$, with $\hat{q}$ the jet quenching parameter (the rate for transverse momentum broadening via interactions in the medium), and $\tau_{\rm br}(z,\omega)$ is the time scale of the branching process. Note that we use light-cone (LC) coordinates and momenta, with the longitudinal axis defined by the direction of motion of the leading particle. Correspondingly, the 'energy' ω truly refers to the LC longitudinal momentum p^+ and t to the LC 'time' x^+. Eq. (1) applies as long as $\ell \ll \tau_{\rm br}(z,\omega) < L$, where L is the length of the medium, and ℓ is the mean free path between successive collisions. The second inequality above implies an upper limit on the average energy of the offspring gluons: $z(1-z)\omega \lesssim \omega_c$, where $\omega_c = \hat{q}L^2/2$ is the maximum energy that can be taken away by a single gluon. It follows from Eq. (1) that the probability for having just one emission throughout the medium is (for z not too close to 1) $\sim \bar{\alpha}\sqrt{\omega_c/z\omega}$, where $\bar{\alpha} \equiv \alpha_s N_c/\pi$. When this becomes of $\mathcal{O}(1)$, i.e, when $z\omega \lesssim \omega_s \equiv \bar{\alpha}^2\omega_c$, multiple branchings become important.

It will be useful to express the energy of a radiated gluon in terms of its energy fraction $x \equiv \omega/E$ and to replace the light-cone time t by the dimensionless variable

$$\tau \equiv \bar{\alpha}\sqrt{\frac{\hat{q}}{E}}\,t = \bar{\alpha}\sqrt{x_c}\,\frac{t}{L}\,, \tag{2}$$

where $x_c \equiv \omega_c/E$. We restrict ourselves here to the case $E < \omega_c$, i.e. $x_c > 1$, leaving the discussion of the $E > \omega_c$ case to a forthcoming publication. Note that the maximal value of τ is $\tau_{\rm max} = \bar{\alpha}\sqrt{x_c}$, corresponding to $t = L$. Then the branching probability (1) can be written as

$$\frac{\mathrm{d}\mathcal{P}_{\rm br}}{\mathrm{d}z\,\mathrm{d}\tau} = \frac{1}{2}\,\frac{\mathcal{K}(z)}{\sqrt{x}}\,, \tag{3}$$

where $\mathcal{K}(z) \equiv f(z)/[z(1-z)]^{3/2} = \mathcal{K}(1-z)$ and $f(z) \equiv \left[1-z(1-z)\right]^{5/2}$.

In this letter, we focus on one observable that characterizes the average properties of the in-medium cascade, namely the gluon spectrum, $D(x,\tau) \equiv x\frac{\mathrm{d}N}{\mathrm{d}x}$. Within the probabilistic picture for multiple branchings, one easily derives the following evolution equation for $D(x,\tau)$

$$\frac{\partial D(x,\tau)}{\partial \tau} = \int \mathrm{d}z\,\mathcal{K}(z)\left[\sqrt{\frac{z}{x}}D\Big(\frac{x}{z},\tau\Big) - \frac{z}{\sqrt{x}}\,D\big(x,\tau\big)\right]. \tag{4}$$

The initial condition corresponds to a single gluon carrying all the energy, that is, $D(x, \tau = 0) = \delta(x-1)$. We shall refer to the right hand side of Eq. (4) as the "collision term", and denote it as $\mathcal{I}[D]$. Its physical interpretation is clear: The first contribution, which is non–local in x (except when x is close to 1), is a *gain term*: it describes the rise in the number of gluons at x due to emissions from gluons at larger x. Note that the function $D(x,\tau)$ has support only for $0 \le x \le 1$, which limits the first z-integral in Eq. (4) to $x < z < 1$. The second contribution to the collision term, local in x, represents a *loss term*, describing the reduction in the number of gluons at x due to their decay into gluons with smaller x. Taken separately, the gain term and the loss term in Eq. (4) have endpoint singularities at $z = 1$, but these singularities exactly cancel in between the two terms and the overall equation is well defined.

For $\tau \ll 1$, we may attempt to solve Eq. (4) in perturbation theory, i.e., by iterations. Thus, by substituting, in the collision term, $D(x,\tau)$ by its initial value $D^{(0)}(x) = \delta(x-1)$, one obtains (for $x < 1$)

$$D^{(1)}(x,\lambda) \,=\, \frac{\tau}{\sqrt{x}(1-x)^{3/2}}. \tag{5}$$

For $\tau = \tau_{max}$, this is just the BDMPSZ spectrum. For reason that will become clear shortly, we refer to the small x part of this spectrum as the "scaling spectrum", i.e., $D_{\rm sc}(x) \sim 1/\sqrt{x}$. A priori, because one expects the small-x region of the spectrum to be populated by multiple branchings, one would expect perturbation theory to break down when $\tau \gtrsim \sqrt{x}$, and consequently the spectrum to be strongly modified in this region. As we shall see, this is not at all the case: the scaling spectrum remains remarkably stable. In fact, if we were to continue solving Eq. (4) by iteration, i.e., inserting the spectrum (5) into the collision term, one would find that after an obvious cancellation in the collision term, the remaining contribution is of the form of the scaling spectrum (see also [7] for a similar observation).

The perturbative analysis turns out to be complicated by the non local character of the gain term (implying a non trivial coupling between the small-x region and the region near $x = 1$). In order to go beyond perturbation theory, and get insight into the non-perturbative features of Eq. (4), we have considered a simpler version of this equation, obtained by modifying the kernel to $\mathcal{K}_0(z) = 1/[z(1-z)]^{3/2}$ (i.e, replacing the smooth function $f(z)$ by 1 in Eq. (3)). This simplification does not affect the singular behavior of the kernel near $z = 0$ and $z = 1$, which determines the qualitative features of the solution, but it allows us to solve Eq. (4) exactly, via a Laplace transform. The solution reads

$$D_0(x,\tau) \,=\, \frac{\tau}{\sqrt{x}(1-x)^{3/2}}\; {\rm e}^{-\pi\frac{\tau^2}{1-x}}\,. \tag{6}$$

The essential singularity at $x = 1$ is a non perturbative effect that can be understood as a Sudakov suppression

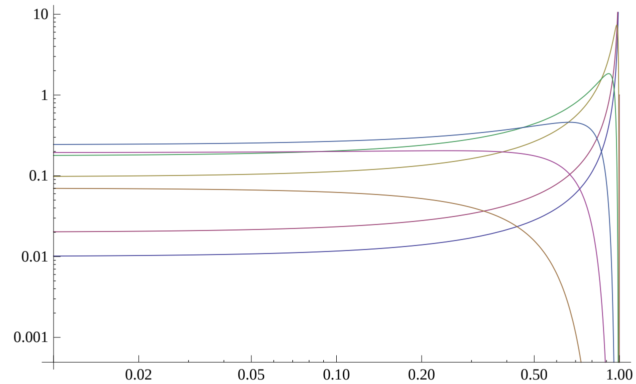

FIG. 1: Plot (in Log-Log scale) of $\sqrt{x}D_0(x,\tau)$, with $D_0(x,\tau)$ given by Eq. (6), as a function of x for various values of τ ($\tau = 0.01, 0.02, 0.1, 0.2, 0.4, 0.6, 0.9$). [*Color online*]

factor [8] (i.e. the vanishing of the probability to emit no gluon in an arbitrary small time). Aside from this exponential factor, one recognizes the scaling spectrum to which $D_0(x,\tau)$ is proportional at small x. This is illustrated in Fig.1: we see that the scaling spectrum is established early on (in agreement with our previous discussion), and remains stable as time progresses. For small times, its amplitude grows linearly with τ: the system can then be viewed as a radiating source located at $x \lesssim 1$, feeding all the small-x modes. As time passes the source weakens and eventually disappears into the left moving "shock wave" visible in Fig. 1. The time dependence of the solution is governed by the factor $\tau \exp[-\pi\tau^2/(1-x)]$ which has an x-dependent maximum: For $x = 0.8$ the maximum occurs at $\tau \simeq 0.18$; this is the time where the peak starts to weaken appreciably, and the shock wave develops. For $x = 0$ the maximum occurs at $\tau = 1/\sqrt{2\pi} \simeq 0.4$; after that time the spectrum decreases exponentially fast. The total amount of energy stored into the modes is given by $\mathcal{E}_0(\tau) = \int_0^1 dx D_0(x,\tau) = {\rm e}^{-\pi\tau^2}$. It is nearly constant for $\tau \lesssim 1/\sqrt{\pi}$, and decreases very rapidly after that. Because the spectrum keeps the same shape as one goes to smaller and smaller x, there is no accumulation of energy at any value $x > 0$. Energy conservation then implies that a 'condensate' develops at $x = 0$, where the excess energy coming from the large x region gets stored. We shall return later to the physical interpretation of this phenomenon.

At this point, it is instructive to analyze an auxiliary problem — that of a system driven by a permanent source of energy localized at $x = 1$. Consider then the equation

$$\frac{\partial D(x,\tau)}{\partial \tau} = A\delta(1-x) + \mathcal{I}[D], \tag{7}$$

For the kernel $\mathcal{K}_0$, one readily verifies that

$$D_{\rm tb}(x,\tau) \,=\, \frac{A}{2\pi\sqrt{x(1-x)}}\,\left(1 - {\rm e}^{-\pi\frac{\tau^2}{1-x}}\right) \tag{8}$$

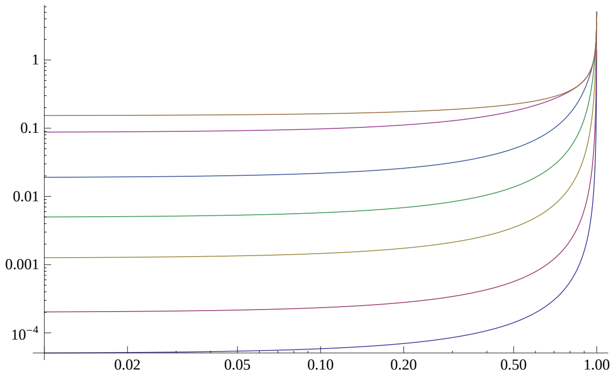

FIG. 2: The function $\sqrt{x}D_{\rm tb}(x,\tau)$ (for $A=1$) at early time, where it resembles Fig. 1, till late time when it approaches the steady state and saturates at the value $A/2\pi$ at small x. The values of τ are: 0.01, 0.02, 0.05, 0.1, 0.2, 0.5, 1.

solves this equation with initial condition $D_{\rm tb}(x,\tau=0)=0$. (Notice that the derivative of $D_{\rm tb}(x,\tau)$ is equal to $D_0(x,\tau)$, to within the multiplicative constant A.) By comparing Fig. 1 and Fig. 2, one sees that the behaviors of $D(x,\tau)$ and $D_{\rm tb}(x,\tau)$ are remarkably similar at small τ. However the most remarkable property of $D_{\rm tb}(x,\tau)$ is that it converges to a time-independent function, $D_{\rm st}(x)=(A/2\pi)/\sqrt{x(1-x)}$. To understand this, we note that $D_{\rm st}(x)$ annihilates (exactly) the collision term, i.e., $\mathcal{I}[D_{\rm st}]=0$, as can be verified by an explicit calculation. As time goes on, the solution $D_{\rm tb}(x,\tau)$ is gradually driven to $D_{\rm st}(x)$, but since this fixed point solution reduces to the scaling spectrum at small x, one observes very little change in this small-x region, except for an overall time-dependent scaling.

The stability of the scaling spectrum, confirmed by this simple calculation, hides important physics that is revealed by calculating the flow of energy that gets transmitted per unit time from the region $x>x_0$ to the region $x<x_0$. If we denote by $\mathcal{E}(x_0,\tau)=\int_{x_0}^1 {\rm d}x D(x,\tau)$ the total energy that is contained in the modes with $x>x_0$ and recognize that the rate of change of $\mathcal{E}(x_0,\tau)$ is due both to a possible source of strength A localized at $x=1$, and to the flow $\mathcal{P}(x_0,\tau)$ at x_0, we get the general expression

$$\mathcal{P}(x_0,\tau)\ \equiv\ A-\frac{\partial\mathcal{E}(x_0,\tau)}{\partial\tau}=-\int_{x_0}^1{\rm d}x\,\mathcal{I}[D]. \qquad (9)$$

An explicit calculation for $D=D_{\rm tb}$ yields

$$\mathcal{P}(x_0,\tau)=A\left[1-{\rm e}^{-\pi\tau^2}{\rm erfc}\left(\sqrt{\frac{\pi x_0}{1-x_0}}\tau\right)\right], \qquad (10)$$

where ${\rm erfc}(x)$ denotes the complementary error function. In order to analyze the physical content of this expression, it is actually useful to rewrite the integral of the collision term in Eq. (9) in the following form

$$\mathcal{P}(x_0,\tau)=\int_0^1{\rm d}z\,\mathcal{K}(z)\int_{x_0}^{\min(1,x_0/z)}{\rm d}x\frac{D\,(x,\tau)}{\sqrt{x}}. \qquad (11)$$

At small times, $\pi\tau^2\ll 1$, and for x_0 not too close to either 0 or 1, one can use the expansion ${\rm erfc}(x)\simeq 1-2x/\sqrt{\pi}$ in Eq. (10) and get $\mathcal{P}(x_0,\tau)\simeq 2A\tau\,\sqrt{x_0/(1-x_0)}$, which vanishes at $x_0=0$: At early times, the energy provided by the source is used to populate the various modes (according to the BDMPSZ spectrum), and does not flow across the entire system. Note that this contribution is obtained from Eq. (11) by substituting $D(x,\tau)$ by $A\tau\delta(1-x)$, as appropriate at small times. This yields indeed $\mathcal{P}(x_0,\tau)\ =\ A\tau\int_0^{x_0}dz\frac{1}{\sqrt{z}(1-z)^{3/2}}\ =\ 2A\tau\sqrt{x_0/(1-x_0)}$. As this last calculation reveals, this early time contribution to the flow involves, when $x_0\ll 1$ very asymmetric branching ($z\leq x_0$). For this reason we interpret it as "direct radiation".

As time goes on however, the distribution of energy among the various modes is such that gain and loss terms equilibrate locally, at which point a steady state is reached with all the energy provided by the source flowing throughout the entire system and leaving the population of the various modes constant. The constant (x_0-independent) contribution to the flow is easily obtained from Eq. (10) by evaluating $\mathcal{P}(x_0,\tau)$ at $x_0=0$, i.e., $\mathcal{P}(x_0=0,\tau)=A(1-{\rm e}^{-\pi\tau^2})$. This result can be recovered from Eq. (11) with $x_0\ll 1$ by approximating $D(x,\tau)$ with the scaling part of the spectrum (8), i.e., $A(1-{\rm e}^{-\pi\tau^2})/(2\pi\sqrt{x})$, and noting that

$$v_0\equiv\int_0^1 dz\frac{1}{\sqrt{z}(1-z)^{3/2}}\ln\frac{1}{z}=2\pi. \qquad (12)$$

What this second calculation demonstrates, is that, in contrast to what happens for the direct radiation, here the typical branchings involve the whole range of z values (about half of the value of the integral (12) comes from the range $0.15\lesssim z\lesssim 0.85$). We refer to this property as "quasi-democratic branching".

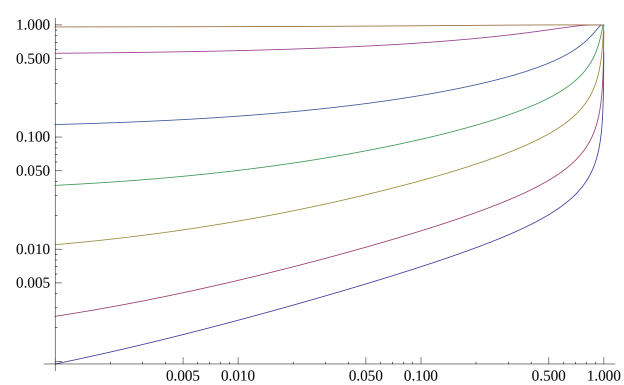

FIG. 3: The flow $\mathcal{P}(x_0,\tau)$ as a function of x_0 from early times to the time where a stationary state is reached ($A=1$). The values of τ are: 0.01, 0.02, 0.05, 0.1, 0.2, 0.5, 1. [*Color online*]

The properties that we have discussed, the existence of a steady solution when the system is coupled to a source, the scaling spectrum whose origin can be traced back to

energy conservation and the particular z-dependence of the kernel $\mathcal{K}(z)$, the presence of a component of the flow that is independent of the position, the quasi-democratic branching, all these are properties that are very reminiscent of *wave turbulence* [10]. Of course, in the present case, the locality of interactions in momentum space, often a crucial ingredient for establishing the existence of steady state solutions, is only marginally satisfied (via what we called quasi-democratic branching). And indeed the separation of the energy flow between radiation and quasi-democratic branching, that we introduced above, is not a clear-cut one; in general both components are mixed in a complicated way. Still these two components are clearly visible in Fig. 3, where we recognize the typical distribution of modes fixed by early time radiation, the nearly uniform flow that develops at small times, and the approach to the steady state at later times.

These properties can be contrasted with those of a typical QCD cascade described by the DGLAP equation. Such an equation can be recovered from Eq. (4) with the substitution $\mathcal{K}(z)/\sqrt{x} \to 1/[z(1-z)]$, where now $\tau = \bar{\alpha}\ln(Q^2/Q_0^2)$, with Q^2 the virtuality. The effect of the evolution is again to redistribute the energy initially concentrated at $x = 1$ towards smaller x. However, in contrast to the case studied in this paper, all the energy of the cascade remains in the spectrum, there is no turbulent flow. In fact it is not difficult to estimate that, in this case, the dominant contribution to the flow is $\mathcal{P}(x_0) \sim x_0 \ln(1/x_0)$ which vanishes when $x_0 \to 0$.

Many of the features that we have uncovered by studying the source problem remain valid without the source, and for the general kernel. We return now to this initial setting. Limiting ourselves to small times, one finds, by calculating the flow from Eq. (11), that it takes the form

$$\mathcal{P}(x_0, \tau) \simeq 2\sqrt{x_0} + \upsilon\tau\,, \tag{13}$$

with $\upsilon = 4.96$ obtained from Eq. (12) with the full $f(z)$ in the numerator. One recognizes the two components that we discussed earlier, that is, the direct radiation and the turbulent flow. Although parametrically subleading (since proportional to τ), the turbulent flow dominates over the direct radiation when $x_0 \lesssim \tau^2$, that is, in the region where multiple branchings are known to be important. The total energy transported by this turbulent flow can be estimated by integrating the second term of Eq. (13) over time. One gets

$$\mathcal{E}_{\rm flow} = E\frac{\upsilon\tau^2}{2} = \frac{\upsilon}{2}\,\bar{\alpha}^2\omega_c. \tag{14}$$

This energy, which is *not* carried by the particles present in the spectrum, is what we identified earlier as the energy stored in a 'condensate' at $x = 0$. In more physical terms, we associate this energy with that transferred to the medium in the form of very soft quanta.

It is beyond the scope of this letter to present a detailed comparison with the data. However the following order-of-magnitude estimates should confirm the relevance of the present discussion for the di–jet asymmetry observed at the LHC. Using the conservative estimate $\omega_c = 60$ GeV (corresponding to $\hat{q} = 1\,\mathrm{GeV}^2/\mathrm{fm}$ and $L \simeq 5$ fm), together with $\bar{\alpha}^2 \simeq 0.1$, one finds $\mathcal{E}_{\rm flow} \simeq 15$ GeV, a value that compares well with the observations. Indeed, the detailed analysis by CMS [3] shows that the energy imbalance between the leading and the subleading jet is compensated by an excess of semi–hard ($p_T < 8$ GeV) quanta propagating at large angles, outside the cone defining the subleading jet. For the most asymmetric events, the total energy in excess is about 25 GeV. Remarkably, most of this energy (about 75%) is carried by very soft quanta with $p_T \leq 2$ GeV. This observation would be difficult to reconcile with the hypothesis that these particles come from gluons in a BDMPSZ spectrum (which would imply that most of the excess energy would be carried by the hardest gluons with energies $\lesssim 8$ GeV). But it could be naturally explained by associating these soft particles with those transported by the turbulent flow that we have discussed in this paper.

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